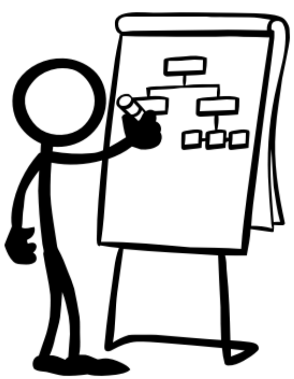
Space Partitions

BSP & Quadtree



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Jan. 24, 2017

Motivation

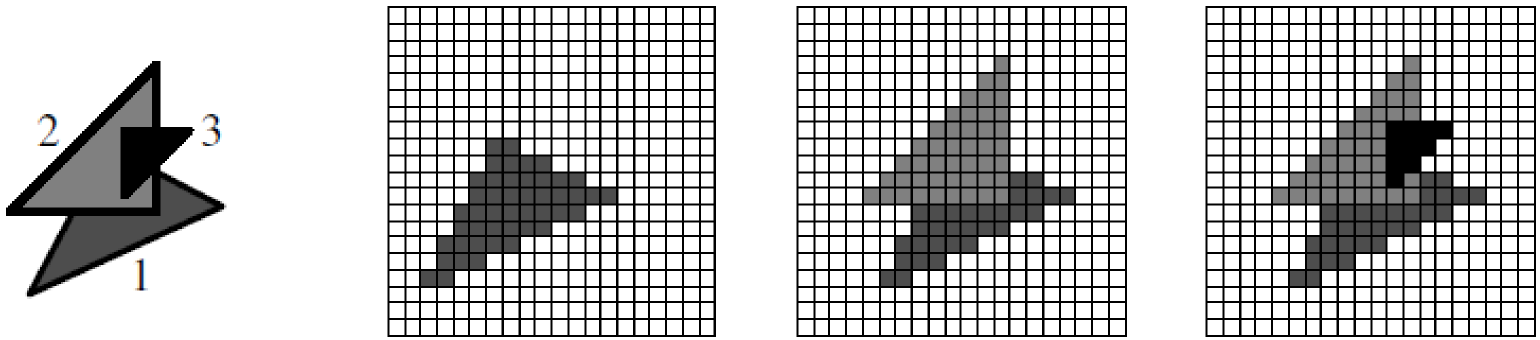
- **Hidden Surface Removal** –
  determine for each pixel on the screen the object that is visible at that pixel

- **z-buffer algorithm**
  - maintains 2 buffers:
    frame buffer stores for each pixel the **intensity** of the currently visible object
    z-buffer stores the **z-coordinate** of the point on the object that is visible at the pixel
  - select a pixel

If z-coordinate of the object at that pixel $<$ the z-coordinate in z-buffer,
frame buffer $\leftarrow$ intensity of the new object
z-buffer $\leftarrow$ z-coordinate

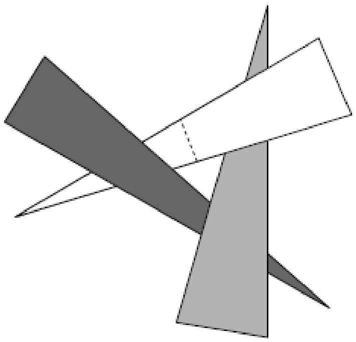
Motivation – z-buffer Alg. vs. Painter's Alg.

- **Disadvantage of z-buffer Algorithm** –
  - Extra storage needed for the z-buffer
  - Extra test on z-coordinate required for every pixel covered by the object
- **Painter's algorithm**
  - Sorting the objects according to their distance to the view point (Avoid extra costs)
  - objects are scan-converted in **depth-order**



Motivation – Problems with Painter's Alg.

- Sort the objects quickly

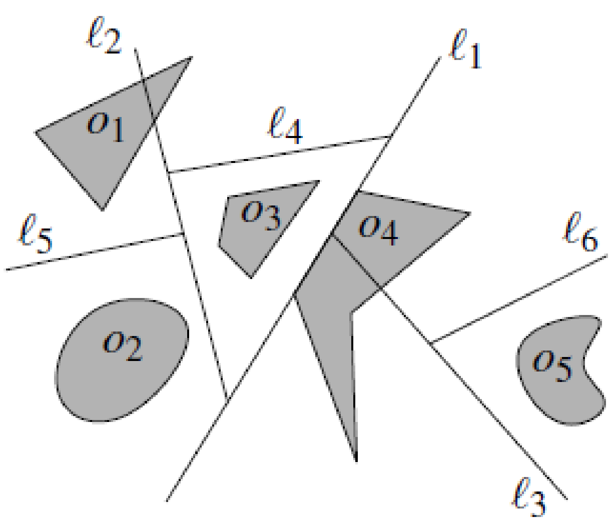


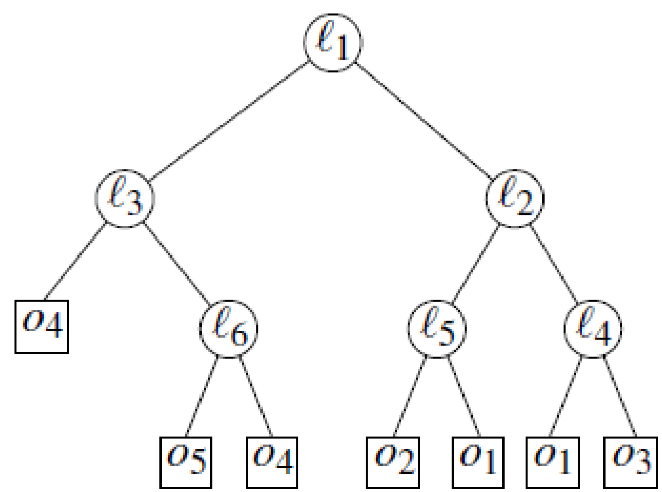
- Depth order may not always exist
  - cyclic overlap
  - **Solutions** – split one or more of the objects till depth order exists for pieces.

- **Binary space partition tree (BSP tree)**

Binary Space Partition (BSP)

- **BSP** is obtained by recursively splitting the plane with a line
  - Splitting lines partition the plane and cut objects into fragments
  - Splitting stops when there is only one fragment in each region





BSP for a set S in $\mathbb{R}^d$ – Definition

- Hyperplane h: $a_1x_1 + a_2x_2 + \cdots + a_dx_d + a_{d+1} = 0$

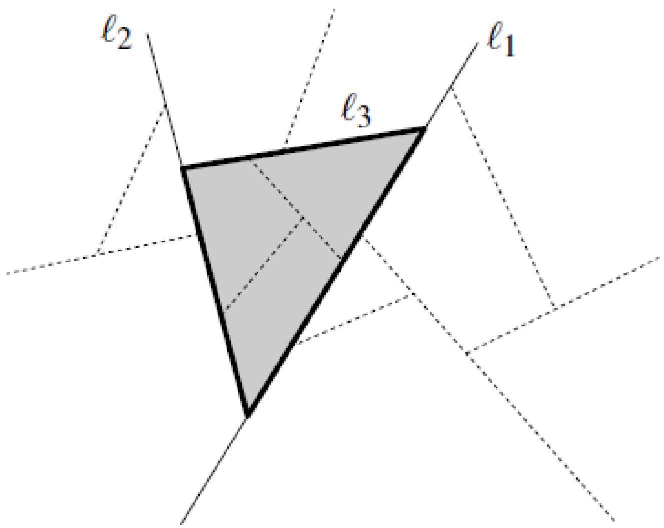
$h^+ := \{(x_1, x_2, \dots, x_d): a_1x_1 + a_2x_2 + \cdots + a_dx_d + a_{d+1} > 0\}$
$h^- := \{(x_1, x_2, \dots, x_d): a_1x_1 + a_2x_2 + \cdots + a_dx_d + a_{d+1} < 0\}$

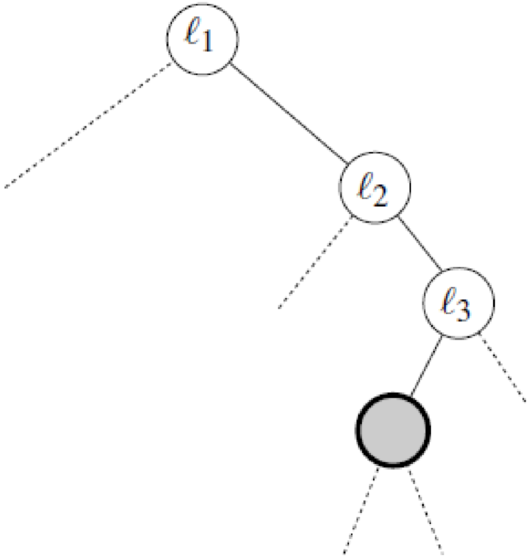
- BSP tree is a defined as a binary tree T with the following properties:

1. If $|S| \leq 1$, T is a leaf. The object fragment in S (if exists) is stored at this leaf.
2. If $|S| \leq 1$, root v of T stores a hyperplane h_v.
 Left child of v is T^- for the set $S^- := \{h_v^- \cap s : s \in S\}$,
 right child of v is T^+ for the set $S^+ := \{h_v^+ \cap s : s \in S\}$.

BSP

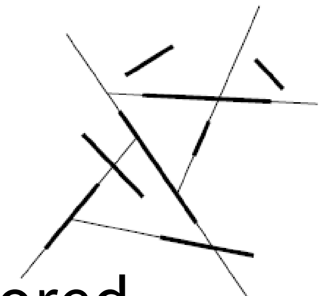
- A node in BSP and its corresponding convex region





BSP for line segments in $\mathbb{R}^2$ - Contruction

- $S = \{s_1, \ldots, s_n\}$ is a set of n non-intersecting line segments in the plane
- Only consider lines containing one of the segments in S as candidate splitting lines **(auto-partitions)**



Algorithm 2DBSP(S)

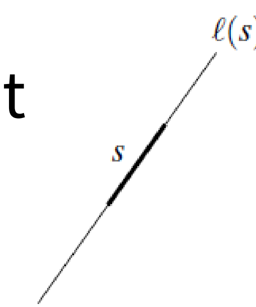
- **If** $|S| \leq 1$

  create T with a single leaf node where S is stored
- **Else**

  $S^- := \{s \cap l(s_1)^- : s \in S\}, T^- \leftarrow \text{2DBSP}(S^-)$

  $S^+ := \{s \cap l(s_1)^+ : s \in S\}, T^+ \leftarrow \text{2DBSP}(S^+)$

  create T with root node v, left subtree T^-, right subtree T^+, and $S(v) = \{s \in S : s \subseteq l(s_1)\}$



Return T

BSP Construction

- Difficult choice ⇒ random choice

Algorithm 2DRandomBSP(S)

- Generate a random permutation $S' = s_1, \ldots, s_n$ of set S
- $T \leftarrow 2\text{DBSP}(S')$
- Return T

- **Lemma:** the expected number of fragments generated by the algorithm 2DRandomBSP is $O(n \log n)$.
- **Proof:**
  - Let s_i be a fixed segment in S
  - Analyze the expected number of other segments that are cut when $l(s_i)$ is added

Proof Continued

- Define the distance of a segment w.r.t. the fixed s_i

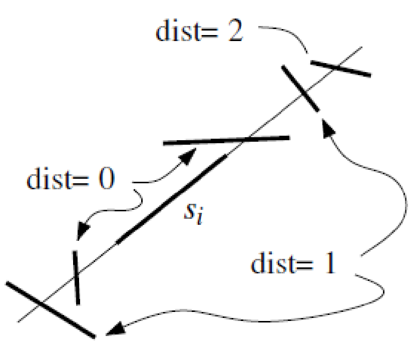
$$\mathrm{dist}_{s_i}(s_j) = \begin{cases} \text{the number of segments intersecting} & \text{if } \ell(s_i) \text{ intersects } s_j \\ \ell(s_i) \text{ in between } s_i \text{ and } s_j & \\ +\infty & \text{otherwise} \end{cases}$$

- Bound the probability that $l(s_i)$ cuts s_j

$$\Pr[\ell(s_i) \text{ cuts } s_j] \leqslant \frac{1}{\mathrm{dist}_{s_i}(s_j)+2}$$

- Bound the expected total number of cuts generated by s_i

$$\begin{aligned} \mathrm{E}[\text{number of cuts generated by } s_i] &\leqslant \sum_{j \neq i} \frac{1}{\mathrm{dist}_{s_i}(s_j)+2} \\ &\leqslant 2\sum_{k=0}^{n-2} \frac{1}{k+2} \\ &\leqslant 2\ln n. \end{aligned}$$



Proof Continued

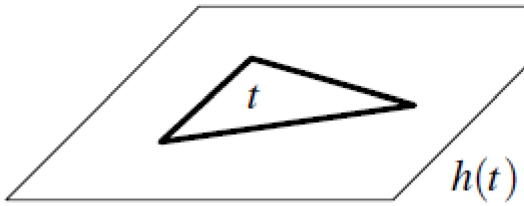
- By linearity of expectation, conclude that the expected total number of cuts generated by all segments is at most $2n \ln n$.

- Expected total number of fragments is bounded by
$$n + 2n \ln n$$

◎ **Theorem:** BSP of size $O(n \log n)$ can be computed in expected time $O(n^2 \log n)$

BSP for triangles in $\mathbb{R}^3$ - Construction

- $S = \{t_1, \dots, t_n\}$is a set of n non-intersecting triangles in $\mathbb{R}^3$
- Only use partition planes containing a triangles of S **(auto-partitions)**



Algorithm 3DBSP(S)

- **If** $|S| \leq 1$

  create T with a single leaf node where S is stored
- **Else**

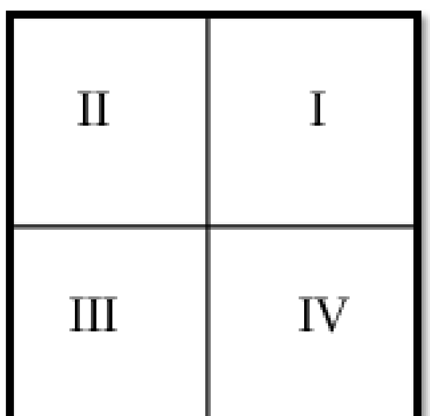
  $S^- := \{t \cap h(t_1)^- : t \in S\}, T^- \leftarrow \text{3DBSP}(S^-)$

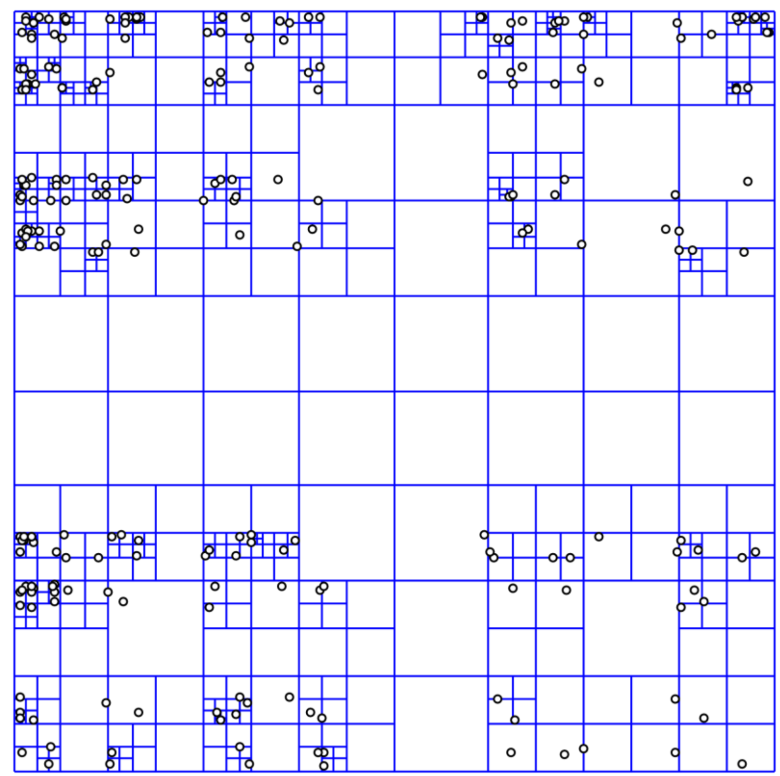
  $S^+ := \{t \cap h(t_1)^+ : t \in S\}, T^+ \leftarrow \text{3DBSP}(S^+)$

  create T with root node v, left subtree T^-, right subtree T^+, and $S(v) = \{t \in S : t \subseteq h(t_1)\}$
- Return T

Quadtree

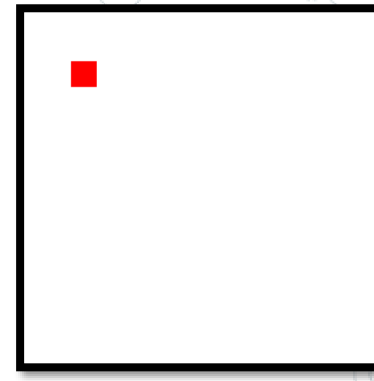
- **Definition** – a tree data structure in which each internal node has exactly four children
- Used to **divide** a 2D region into more manageable parts
- **Nodes** – axis-aligned squares



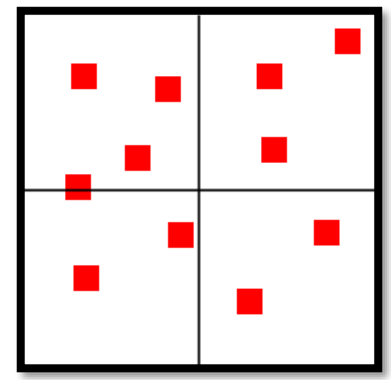


Quadtree

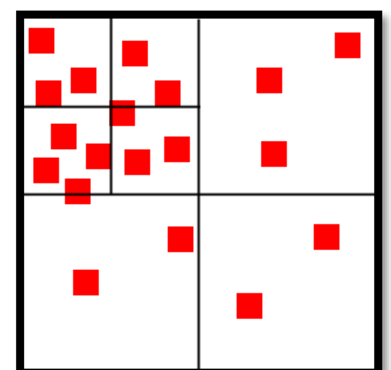
- Starts as a single node



- Splits into 4 subnodes when more objects are added
- object that cannot fully fit inside a node's boundary will be placed in the parent node



- Continue subdividing till the number of objects in each cell is $O(1)$



Depth of Quadtrees of Point Sets

- **Lemma:** the depth of a quadtree of point set S with minimal distance c and bounding box of side length s is at most $\log\left(\frac{s}{c}\right) + \frac{3}{2}$.
- **Proof:**
  - Side length of a square at depth i is $\frac{s}{2^i}$
  - Maximum distance between 2 points inside a square is the length of the diagonal, $\frac{\sqrt{2}s}{2^i}$
  - An internal node at the 'second last level' has at least 2 points, denote its depth d

Proof - Continue

- Internal node at depth i must satisfy:
  $$\frac{\sqrt{2s}}{2^d} \geq c \Rightarrow d \leq \log_2\left(\frac{\sqrt{2s}}{c}\right) = \log_2\left(\frac{s}{c}\right) + \frac{1}{2}$$
- Depth of leaf is at most $d + 1$.

◎ If $s \gg c$, the tree is far from being balanced